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Schwinger Pair Production via Instantons in a Strong Electric Field

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Abstract

In the space-dependent gauge, each mode of the Klein-Gordon equation in a strong electric field takes the form of a time-independent Schrödinger equation with a potential barrier. We propose that the single- and multi-instantons of quantum tunneling may be related with the single- and multi-pair production of bosons and the relative probability for the no-pair production is determined by the total tunneling probability via instantons. In the case of a uniform electric field, the instanton interpretation recovers exactly the well-known pair production rate for bosons and when the Pauli blocking is taken into account, it gives the correct fermion production rate. The instanton is used to calculate the pair production rate even in an inhomogeneous electric field. Furthermore, the instanton interpretation confirms the fact that bosons and fermions can not be produced by a static magnetic field only.

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I. INTRODUCTION

A strong electromagnetic field leads to two physically important phenomena. The effective action due to the vacuum fluctuations of an external electromagnetic field results in the nonlinear Maxwell equations [1]. A strong electric field leads to the significant pair production of bosons and fermions due to the vacuum instability [2]. As its long history, there have been developed many different methods to derive the QED effective action in external electromagnetic fields and to apply to various physical problems.

The proper time method by Schwinger [2] and DeWitt [3] has widely been employed to compute the effective action for bosons and fermions. The real part of the effective action leads to the vacuum polarization and the imaginary part to the pair production. Though that method is conceptually well-defined and technically rigorous, it is sometimes difficult to apply the method to some concrete physical problems such as inhomogeneous electromagnetic fields and others. A canonical method has also been used in many physical contexts [3]. The canonical method proves quite efficient in calculating the pair production rate of bosons and fermions in a strong uniform or inhomogeneous electric field.

In the canonical method the most frequently used gauge of an electromagnetic potential is the time-dependent gauge. In that gauge the Klein-Gordon equation for bosons or the Dirac equation for fermions in an external electric field, when appropriately mode-decomposed, takes the form of time-dependent Schrödinger equations. Now the pair production by the external electric field is analogous to the particle production by a time-dependent metric of a curved spacetime [4]. In both problems one imposes the same boundary condition on the solution that an incident positive frequency component in the past infinity is scattered by a potential barrier into the superposition of a positive and a negative frequency component in the future infinity. The boundary condition is the complex conjugate of that of a scattering problem in quantum mechanics. The coefficients determine the Bogoliubov transformation and, in particular, the coefficient of the negative frequency component gives the number of bosons or fermions per mode produced by the electric field [5,6].

A shortcoming of the time-dependent gauge is that except for a uniform field, the gauge potential and thereby the Klein-Gordon equation involve both the space and time coordinates at the same time. So one may not apply directly the above interpretation of particle production via the Bogoliubov transformation to each mode of the Klein-Gordon equation. On the other hand, in the space-dependent (Coulomb) gauge, upon an appropriate mode-decomposition, each mode of the Klein-Gordon equation for bosons or the Dirac equation for fermions takes the form of a time-independent Schrödinger equation for quantum tunneling through a potential barrier. In that space-dependent gauge there is no direct interpretation of wave components in terms of a positive and a negative frequency. Padmanabhan [6] suggested that the probability for reflection from the standard boundary condition for the scattering problem gives the correct relative probability for the vacuum-to-vacuum transition of boson. Popov [7] and Brout *et al* [8] also noticed the role of instanton of the transmitted wave in describing pair production.

It is the purpose of this paper to interpret in terms of instantons the pair production of bosons and fermions in a strong electric field in any spacetime dimensions. In the space-dependent (Coulomb) gauge we propose that the single- and multi-instantons for quantum tunneling determine somehow the single- and multi-pair production for each mode and show that all the contributions from multi-instantons and anti-instantons yield exactly the total tunneling probability and therefrom determine the relative vacuum-to-vacuum transition, from which the boson pair production is calculated. We further show that the instanton interpretation together with the Pauli blocking gives correctly the production rate for fermions per unit volume and time. The pair production rates of bosons and fermions are calculated using WKB (adiabatic) approximation for the instantons in an inhomogeneous electric field. Finally we show that according to the instanton interpretation a static localized magnetic field does not lead to any pair production, confirming the result from the proper time method.

The organization of this paper is as follows. In Sec. II we show that the tunneling probability by instanton gives correctly the relative rates for the pair production of bosons and fermions in a uniform electric field. We calculate the pair production rates in any spacetime dimensions and compare them with those from other methods. In Sec. III we extend the instanton interpretation of pair production to an inhomogeneous electric field and find the pair production rates in terms of instantons. In Sec. IV we apply the idea to a static magnetic field and show no-pair production in this case. This resolves some of a puzzling issue on the pair production by a static localized magnetic field.

II. UNIFORM ELECTRIC FIELD

We consider a charged boson in a constant electric field in a $(d+1)$-dimensional Minkowski spacetime. It satisfies the Klein-Gordon equation (in units $\hbar = c = 1$)

$$\left[\eta^{\mu\nu}\left(\frac{\partial}{\partial x^{\mu}} + iqA_{\mu}\right)\left(\frac{\partial}{\partial x^{\nu}} + iqA_{\nu}\right) + m^2\right]\Phi(t, \mathbf{x}) = 0, \tag{1}$$

where q is the charge and m the mass of the boson. In the space-dependent (Coulomb) gauge, the vector potential for the constant electric field in the $x_{\parallel}$-direction is given by

$$A_{\mu}(t, \mathbf{x}) = (-E_0 x_{\parallel}, 0, \cdots, 0). \tag{2}$$

When the boson field is mode-decomposed as

$$\Phi(t, \mathbf{x}) = e^{i(\mathbf{k}_{\perp}\cdot\mathbf{x}_{\perp} - \omega t)}\phi_{\omega,\mathbf{k}_{\perp}}(x_{\parallel}), \tag{3}$$

each mode of the Klein-Gordon equation satisfies a one-dimensional Schrödinger-like equation

$$\left[-\frac{1}{2}\frac{d^2}{dx_{\parallel}^2} - \frac{1}{2}\left(\omega + qE_0 x_{\parallel}\right)^2\right]\phi_{\omega,\mathbf{k}_{\perp}}(x_{\parallel}) = -\frac{1}{2}(m^2 + \mathbf{k}_{\perp}^2)\phi_{\omega,\mathbf{k}_{\perp}}(x_{\parallel}). \tag{4}$$

One may interpret Eq. (4) as a quantum system of a unit mass having the inverted harmonic potential with the center at $x_{\parallel,c} = -\frac{\omega}{qE_0}$ and the energy $\epsilon = -\frac{1}{2}(m^2 + \mathbf{k}_{\perp}^2)$. Due to the negative energy $\epsilon < 0$, Eq. (4) describes a tunneling problem. By introducing the dimensionless variable and parameter

$$\xi = \sqrt{\frac{2}{qE_0}}(\omega + qE_0 x_{\parallel}), \quad a_{\mathbf{k}_{\perp}} = \frac{m^2 + \mathbf{k}_{\perp}^2}{2qE_0}, \tag{5}$$

Eq. (4) can be rewritten as

$$\left[\frac{d^2}{d\xi^2}+\frac{1}{4}\xi^2-a_{\mathbf{k}_\perp}\right]\phi_{\omega,\mathbf{k}_\perp}(\xi)=0. \tag{6}$$

The wave function describing tunneling process is given by the complex parabolic cylindrical function [9]

$$\phi_{\omega,\mathbf{k}_\perp}(\xi)=cE(a_{\mathbf{k}_\perp},\xi), \tag{7}$$

where c is a complex number. It has the asymptotic forms

$$\begin{aligned}
\phi_{\omega,\mathbf{k}_\perp}(\xi) &= A\sqrt{\frac{2}{|\xi|}}e^{-i\frac{\xi^2}{4}}-B\sqrt{\frac{2}{|\xi|}}e^{i\frac{\xi^2}{4}}, \quad (\xi\ll -2\sqrt{a_{\mathbf{k}_\perp}}),\\
\phi_{\omega,\mathbf{k}_\perp}(\xi) &= C\sqrt{\frac{2}{\xi}}e^{i\frac{1}{4}\xi^2}, \quad (\xi\gg 2\sqrt{a_{\mathbf{k}_\perp}}),
\end{aligned} \tag{8}$$

where

$$A=ic\sqrt{1+e^{2\pi a_{\mathbf{k}_\perp}}}, \quad B=-ice^{\pi a_{\mathbf{k}_\perp}}, \quad C=c. \tag{9}$$

In the space-dependent gauge, Padmanabhan interpreted that the reflection probability of the wave function gives the relative probability for the vacuum-to-vacuum transition [6]. His interpretation implies that an incoming (initial) vacuum state is described by $\phi_{\omega,\mathbf{k}_\perp}(x_\parallel=-\infty)$ and thus the reflected wave function denotes an outgoing (final) vacuum state. We further propose that the total tunneling probability $P^{\rm tun.}$ via the single- and multi-instantons represents in a certain way the single- and multi-pair production of bosons and the relative probability for the vacuum-to-vacuum transition is given by the probability for the no-pair production $P^{\rm no-pair}=1-P^{\rm tun.}$, *i.e*, the reflection probability. It should be noted that the instanton interpretation for pair production excludes a non-zero transmission probability above a potential barrier or a potential well from the tunneling probability via instantons and thus implies no-pair production.

To see how the instanton interpretation works for the uniform electric field, we calculate the tunneling probability from the asymptotic form (8) of wave function and compare it with the result from instanton calculation. The total tunneling probability for each mode $\mathbf{k}_\perp$ is given by

$$P^{\rm b.\ tun.}_{\mathbf{k}_\perp}=\left|\frac{C}{A}\right|^2=\frac{1}{e^{2\pi a_{\mathbf{k}_\perp}}+1}, \tag{10}$$

and the probability for the no-pair production, *i.e*, the vacuum-to-vacuum transition, by

$$P^{\rm b.\ no-pair}_{\mathbf{k}_\perp}=\left|\frac{B}{A}\right|^2=\frac{1}{1+e^{-2\pi a_{\mathbf{k}_\perp}}}. \tag{11}$$

In fact, it is the consequence of flux conservation in quantum mechanics that Eq. (11) is determined by Eq. (10) as

$$P_{\mathbf{k}_\perp}^{\text{b. no-pair}} = 1 - P_{\mathbf{k}_\perp}^{\text{b. tun.}}. \tag{12}$$

Hence, what one needs in finding the probability for the no-pair production (vacuum-to-vacuum transition) even in a general electric field is the corresponding total tunneling probability via the single- and multi-instantons.

Now the tunneling probability (10) can be interpreted in terms of multi-instantons and anti-instantons of tunneling process. In instanton physics [10], the leading contribution to the tunneling probability

$$P_{\mathbf{k}_\perp}^{\text{tun.}} = e^{-2S_{\mathbf{k}_\perp}}, \tag{13}$$

is determined by the single-instanton action

$$S_{\mathbf{k}_\perp} = \int_{x_-}^{x^+} dx_\parallel \sqrt{m^2 + \mathbf{k}_\perp^2 - \Big(\omega + qE_0 x_\parallel\Big)^2} = \pi a_{\mathbf{k}_\perp}, \tag{14}$$

where $x_\pm = \pm\sqrt{m^2 + \mathbf{k}_\perp^2} - \omega$ are the classical turning points. The single-instanton may be related in a certain way with one-pair production, and the multi-instantons with multi-pair production. As there is no limitation from the Pauli blocking for the multi-pair production of bosons, multi-instantons lead to the multi-pair production, whereas multi-anti-instantons lead to the annihilation of created boson pair and therefore contribute to the no-pair production (vacuum-to-vacuum transition). So the correct total tunneling probability should take into account both multi-instantons and anti-instantons

$$P_{\mathbf{k}_\perp}^{\text{b. tun.}} = \sum_{n=1}^{\infty} (-1)^{n+1} e^{-2nS_{\mathbf{k}_\perp}} = \frac{1}{e^{2S_{\mathbf{k}_\perp}} + 1}, \tag{15}$$

where instantons contribute positively and anti-instantons negatively. Similarly the relative probability for the no-pair production (vacuum-vacuum transition) is given by

$$P_{\mathbf{k}_\perp}^{\text{b. no-pair}} = \sum_{n=0}^{\infty} (-1)^{n} e^{-2nS_{\mathbf{k}_\perp}} = \frac{1}{1 + e^{-2S_{\mathbf{k}_\perp}}}, \tag{16}$$

These results agree with Eqs. (10) and (11). The physical interpretation of the alternating signs is that only instantons of even times repeated periodic motions in the inverted potential contribute positively (the right hand side) to the tunneling, whereas anti-instantons of odd times repeated periodic motions contribute negatively (the left hand side) to the tunneling.

The vacuum means the absence of any particle for possible physical states. So the vacuum-persistence, *i.e.*, the vacuum-to-vacuum transition, is the total relative probability for the no-pair production:

$$|\langle 0, \text{out}|0, \text{in}\rangle|^2 = \prod_{\text{all states}} P_{\mathbf{k}_\perp}^{\text{b. no-pair}} = \exp\Big[- \sum_{\text{all states}} \ln(1 + e^{-2S_{\mathbf{k}_\perp}})\Big]. \tag{17}$$

On the other hand, the vacuum persistence is given by the imaginary part of the effective action

$$|\langle 0, \text{out}|0, \text{in}\rangle|^2 = \exp\Big[-2VT \text{Im}\mathcal{L}_{\text{eff.}}^{\text{boson}}\Big], \tag{18}$$

where V and T are the relevant volume and duration of time. Therefore, the pair production rate per unit volume and time is given by the imaginary part of the effective action:

$$\begin{aligned} \mathrm{Im}\mathcal{L}_{\mathrm{eff.}}^{\mathrm{boson}} &= \frac{1}{2VT}\sum_{\mathrm{all\ states}} \ln(1+e^{-2S_{\mathbf{k}_\perp}}) \\ &= \frac{(2s+1)}{2T}\int \frac{d\mathbf{k}^d}{(2\pi)^d}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{\pi n}{qE_0}\mathbf{k}_\perp^2}e^{-\frac{\pi m^2}{qE_0}n} \\ &= \frac{(2s+1)}{2(2\pi)^d}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{qE_0}{n}\right)^{(d+1)/2}e^{-\frac{\pi m^2}{qE_0}n}, \end{aligned} \tag{19}$$

where s is the spin of the boson, and we used $\int dk_\parallel = (qE_0)T$ [11]. It should be noted that Eq. (19) recovers the standard result for the boson pair production in Ref. [12].

The fermion pair production can be understood similarly. The created fermion pair blocks the multi-pair production. So the total tunneling probability related with the fermion pair production per each mode is just

$$P_{\mathbf{k}_\perp}^{\mathrm{f.\ tun.}} = e^{-2S_{\mathbf{k}_\perp}}. \tag{20}$$

Therefore, the relative probability for the no-pair production of fermions is given by

$$P_{\mathbf{k}_\perp}^{\mathrm{f.\ no-pair}} = 1 - e^{-2S_{\mathbf{k}_\perp}}. \tag{21}$$

Finally, the fermion pair production rate per unit volume and time is found to be

$$\begin{aligned} \mathrm{Im}\mathcal{L}_{\mathrm{eff.}}^{\mathrm{fermion}} &= -\frac{1}{2VT}\sum_{\mathrm{all\ states}} \ln(1-e^{-2S_{\mathbf{k}_\perp}}) \\ &= \frac{(2s+1)}{2T}\int \frac{d\mathbf{k}^d}{(2\pi)^d}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\frac{\pi n}{qE_0}\mathbf{k}_\perp^2}e^{-\frac{\pi m^2}{qE_0}n} \\ &= \frac{(2s+1)}{2(2\pi)^d}\sum_{n=1}^{\infty}\left(\frac{qE_0}{n}\right)^{(d+1)/2}e^{-\frac{\pi m^2}{qE_0}n}. \end{aligned} \tag{22}$$

Equation (22) also recovers the standard result for the fermion pair production in Ref. [12].

III. INHOMOGENEOUS ELECTRIC FIELDS

We now consider the pair production in a static inhomogeneous electric field. For the sake of simplicity, the electric field is assumed to be applied in the $x_\parallel$-direction and to have the gauge potential

$$A_\mu(t,\mathbf{x}) = (A_0(x_\parallel), 0, \cdots, 0), \tag{23}$$

where $E(x_\parallel) = -\frac{dA_0(x_\parallel)}{dx_\parallel}$. Then the mode-decomposed Klein-Gordon equation takes the form

$$\left[-\frac{1}{2}\frac{d^2}{dx_\parallel^2} - \frac{1}{2}\left(\omega - qA_0(x_\parallel)\right)^2\right]\phi_{\omega,\mathbf{k}_\perp}(x_\parallel) = -\frac{1}{2}(m^2+\mathbf{k}_\perp^2)\phi_{\omega,\mathbf{k}_\perp}(x_\parallel). \tag{24}$$

We can still interpret Eq. (24) as a one-dimensional quantum system of a unit mass with the potential $-\frac{1}{2}(\omega - qA_0(x_\parallel))^2$ and the energy $\epsilon = -\frac{1}{2}(m^2 + \mathbf{k}_\perp^2)$. In the WKB (adiabatic) approximation the asymptotic form for the tunneling probability for each mode $\mathbf{k}_\perp$ is given by [13]

$$P_{\mathbf{k}_\perp}^{\text{b. tun.}} = \frac{1}{e^{2S_{\mathbf{k}_\perp}} + 1}, \tag{25}$$

where

$$S_{\mathbf{k}_\perp} = \sum_{n=0}^{\infty} S_{\mathbf{k}_\perp}^{(2n)}. \tag{26}$$

Here the leading contribution to $S_{\mathbf{k}_\perp}$ is given by the instanton

$$S_{\mathbf{k}_\perp}^{(0)} = \oint dx_\parallel \Big[Q_{\mathbf{k}_\perp}(x)\Big]^{1/2}, \tag{27}$$

and the next leading term by

$$S_{\mathbf{k}_\perp}^{(2)} = \oint dx_\parallel \left[\frac{1}{8}\frac{Q''_{\mathbf{k}_\perp}(x)}{Q^{3/2}_{\mathbf{k}_\perp}(x)} - \frac{5}{32}\frac{Q'^2_{\mathbf{k}_\perp}(x)}{Q^{5/2}_{\mathbf{k}_\perp}(x)}\right], \tag{28}$$

where

$$Q_{\mathbf{k}_\perp}(x) = m^2 + \mathbf{k}_\perp^2 - \Big(\omega - qA_0(x_\parallel)\Big)^2. \tag{29}$$

Hence the relative probability for the no-pair production (vacuum-vacuum transition) of bosons is given by

$$P_{\mathbf{k}_\perp}^{\text{b. no-pair}} = \frac{1}{1 + e^{-2S_{\mathbf{k}_\perp}}}, \tag{30}$$

and for fermions by

$$P_{\mathbf{k}_\perp}^{\text{f. no-pair}} = 1 - e^{-2S_{\mathbf{k}_\perp}}. \tag{31}$$

A few comments are in order. First, if the electric field extends over all the space and has the potential $|A_0(\pm\infty)| = \infty$, then the potential barrier decreases indefinitely at both $\pm\infty$. Therefore, there are instantons for all $\mathbf{k}_\perp$. Second, if the electric field is localized or has finite values of the potential at $\pm\infty$, then only those modes belonging to $k_{\perp,+} \geq |\mathbf{k}_\perp| \geq k_{\perp,-}$ lead to pair production, where the upper and lower limits are given by

$$\mathbf{k}_{\perp,\pm}^2 = \Big(\omega - qA_0(\pm\infty)\Big)^2 - m^2. \tag{32}$$

In the inhomogeneous electric field, we obtain the boson pair production rate per unit volume and time

$$\begin{aligned}\mathrm{Im}\mathcal{L}^{\rm boson}_{\rm eff.} &= \frac{(2s+1)}{2VT}\sum_{\rm all\ allowed\ states}\ln\Bigl(1+e^{-2S_{\mathbf{k}_\perp}}\Bigr)\\ &= \frac{(2s+1)}{2(2\pi)^d}\frac{(d-1)\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_{k_{\perp,-}}^{k_{\perp,+}}dk_\perp k_\perp^{d-2}e^{-2nS_{\mathbf{k}_\perp}},\end{aligned}\tag{33}$$

and the fermion pair production rate

$$\begin{aligned}\mathrm{Im}\mathcal{L}^{\rm fermion}_{\rm eff.} &= -\frac{(2s+1)}{2VT}\sum_{\rm all\ allowed\ states}\ln\Bigl(1-e^{-2S_{\mathbf{k}_\perp}}\Bigr)\\ &= \frac{(2s+1)}{2(2\pi)^d}\frac{(d-1)\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})}\sum_{n=1}^{\infty}\frac{1}{n}\int_{k_{\perp,-}}^{k_{\perp,+}}dk_\perp k_\perp^{d-2}e^{-2nS_{\mathbf{k}_\perp}}.\end{aligned}\tag{34}$$

As an exactly solvable model we consider a localized electric field $E(x_\parallel) = E_0 \mathrm{sech}^2(\frac{x_\parallel}{L})$ with the Sauter type gauge potential [11]

$$A_0(x_\parallel) = -E_0 L \tanh(\frac{x_\parallel}{L}).\tag{35}$$

In the limit of $L \to \infty$ the gauge potential (35) reduces to the uniform electric field in Sec. II. Since the gauge potential (35) is a more general case including the uniform field as a special case, it is worthy to apply the instanton interpretation to pair production and compare the result with the exact one. Bosons gain an additional contribution to momenta from the acceleration by the localized electric field and have asymptotic values at $x_\parallel \to \pm\infty$:

$$k_\parallel^2(\infty) = (qE_0L+\omega)^2 - m^2 - \mathbf{k}_\perp^2, \quad k_\parallel^2(-\infty) = (qE_0L-\omega)^2 - m^2 - \mathbf{k}_\perp^2.\tag{36}$$

In the large L limit the instanton action (27) is given by

$$S_{\mathbf{k}_\perp} = \pi\frac{m^2+\mathbf{k}_\perp^2}{qE_0}\Biggl[1+\frac{\omega^2}{q^2E_0^2L^2}+\frac{m^2+\mathbf{k}_\perp^2}{8q^2E_0^2L^2}+\mathcal{O}(\frac{1}{L^4})\Biggr].\tag{37}$$

The exact wave function describing the tunneling process is found

$$\phi_{\omega,\mathbf{k}_\perp}(x_\parallel) = Ce^{-\mu\frac{x_\parallel}{L}}\mathrm{sech}^\nu(\frac{x_\parallel}{L})F(\alpha,\beta;\gamma;\zeta),\tag{38}$$

where F is the hypergeometric function and

$$\begin{aligned}&\mu = -i\frac{L}{2}\Bigl(k_\parallel(\infty)+k_\parallel(-\infty)\Bigr), \quad \nu = -i\frac{L}{2}\Bigl(k_\parallel(\infty)-k_\parallel(-\infty)\Bigr),\\ &\alpha = \nu+\frac{1}{2}+i\sqrt{(qE_0L^2)^2-\frac{1}{4}}, \quad \beta = \nu+\frac{1}{2}-i\sqrt{(qE_0L^2)^2-\frac{1}{4}},\\ &\gamma = 1-iLk_\parallel(\infty), \quad \zeta = \frac{1}{2}\Bigl(1-\tanh(\frac{x_\parallel}{L})\Bigr).\end{aligned}\tag{39}$$

In the limit of $x_\parallel \gg L$, Eq. (38) has the asymptotic form

$$\phi_{\omega,\mathbf{k}_\perp}(x_\parallel) = 2^\nu C e^{ik_\parallel(\infty)x_\parallel}.\tag{40}$$

It describes a wave function after tunneling. In the limit of $x_\parallel \ll -L$, we may use another form for Eq. (38)

$$\phi_{\omega,\mathbf{k}_\perp}(x_\parallel) = Ce^{-\mu\frac{x_\parallel}{L}}\mathrm{sech}^\nu(\frac{x_\parallel}{L})\Bigg[\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta;\gamma;1-\zeta) + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-\zeta)^{\gamma-\alpha-\beta}F(\alpha,\beta;\gamma;1-\zeta)\Bigg]. \tag{41}$$

In that limit the first term of Eq. (41) describes an incident wave having the asymptotic form

$$\phi_{\omega,\mathbf{k}_\perp}(x_\parallel) = 2^\nu C\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}e^{ik_\parallel(-\infty)x_\parallel}. \tag{42}$$

Therefore, from Eqs. (40) and (42) we can find the probability for tunneling

$$\begin{aligned} P^{\text{b. tun.}}_{\mathrm{k}_\perp} &= \frac{k_\parallel(-\infty)}{k_\parallel(\infty)}\left|\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}\right|^2 \\ &= \frac{\sinh\pi\Big(Lk_\parallel(\infty)\Big)\sinh\pi\Big(Lk_\parallel(-\infty)\Big)}{\cosh\pi\Big(\frac{L}{2}(k_\parallel(\infty)+k_\parallel(-\infty))+Q\Big)\cosh\pi\Big(\frac{L}{2}(k_\parallel(\infty)+k_\parallel(-\infty))-Q\Big)}, \end{aligned} \tag{43}$$

where $Q=\sqrt{(qE_0L^2)^2-\frac{1}{4}}$. In the large L limit we obtain approximately the probability for tunneling

$$P^{\text{b. tun.}}_{\mathrm{k}_\perp} = \frac{1}{1+e^{2S_{\mathbf{k}_\perp}}}. \tag{44}$$

Here, we used the binomial expansion

$$\begin{aligned} k_\parallel(\pm\infty) &= (qE_0L\pm\omega)\left[1-\frac{m^2+\mathbf{k}_\perp^2}{(qE_0L)^2\Big(1\pm\frac{\omega}{(qE_0L)^2}\Big)^2}\right]^{1/2} \\ &= qE_0L\pm\omega-\frac{m^2+\mathbf{k}_\perp^2}{2qE_0L}\left[1\mp\frac{\omega}{qE_0L}+\frac{\omega^2}{(qE_0L)^2}\right] \\ &\quad -\frac{(m^2+\mathbf{k}_\perp^2)^2}{8(qE_0L)^3}\left[1\mp\frac{3\omega}{qE_0L}+\frac{6\omega^2}{(qE_0L)^2}\right]+\cdots. \end{aligned} \tag{45}$$

Therefore, using instanton action (37) one can obtain the pair production rate for bosons and fermions according to Eqs. (33) and (34). Thus we have shown that in the space-dependent gauge the instanton interpretation for wave function gives correctly the pair production rates for bosons and fermions for two exactly solvable models.

IV. MAGNETIC FIELDS

Recently the possibility of pair production by a static localized magnetic field has been raised in Ref. [14]. In this section we resolve this issue from the view point of the instanton interpretation.

Let us consider a static magnetic field in a 4-dimensional spacetime with the gauge potential

$$A_\mu(t, \mathbf{x}) = (0, A_1(x_2), 0, 0). \tag{46}$$

The magnetic field is given by $\mathbf{B} = \frac{dA_1(x_2)}{dx_2}\hat{\mathbf{x}}_3$. The Klein-Gordon equation has the form

$$\left[\frac{\partial^2}{\partial t^2} - \left(\frac{\partial}{\partial x_1} + iqA_1(x_2)\right)^2 - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + m^2\right]\Phi(t, \mathbf{x}) = 0. \tag{47}$$

As in the case of the electric field, each mode of the field

$$\Phi(t, \mathbf{x}) = e^{i(k_1 x_1 + k_3 z_3 - \omega t)}\phi_{\omega,k_1,k_3}(x_2) \tag{48}$$

leads to a Schrödinger-like equation

$$\left[-\frac{1}{2}\frac{d^2}{dx_2^2} + \frac{1}{2}\left(k_1 - qA_1(x_2)\right)^2\right]\phi_{\omega,k_1,k_3}(x_2) = \frac{1}{2}(\omega^2 - m^2 - k_3^2)\phi_{\omega,k_1,k_3}(x_2). \tag{49}$$

As a one-dimensional quantum system, Eq. (49) has the potential $\frac{1}{2}(k_1 - qA_1(x_2))^2$ and the energy $\frac{1}{2}(\omega^2 - m^2 - k_3^2)$.

In the case of a uniform magnetic field, the gauge potential $A_1(x_2) = -B_0 x_2$ is indefinitely unbounded at $x_2 = \pm\infty$. Then the potential of Eq. (49) is exactly that of a harmonic oscillator and the energy is quantized

$$\epsilon_n = qB_0(2n + 1). \tag{50}$$

The quantized energy has been used to calculate the effective action in the uniform magnetic field [1]. From the view point of instanton interpretation, there is no pair production since there are not instantons at all. This result agrees with that obtained from the proper time method and other methods.

We now consider a localized magnetic field $\mathbf{B}(x_2) = B_0 \mathrm{sech}^2\left(\frac{x_2}{L}\right)\hat{\mathbf{x}}_3$. The gauge potential is given by $A_1(y) = -B_0 L \tanh\left(\frac{x_2}{L}\right)$. The gauge potential in Eq. (49) has two asymptotic values $\frac{1}{2}(k_1 \pm qB_0 L)^2$ at $x_2 = \pm\infty$, respectively, and a minimum value in-between. So there is no potential barrier for instantons to exist. Therefore, according to the instanton interpretation, there is no pair production, and this fact contradicts with the result in Ref. [14].

V. DISCUSSION AND CONCLUSION

In this paper we have studied the pair production of bosons and fermions by a strong electric field. We have employed the canonical method to find the wave functions of the Klein-Gordon equation and applied an appropriate boundary condition. In the space-dependent (Coulomb) gauge the task of calculating pair production reduces to that of finding tunneling probability via instantons. The boundary condition for the case of space-dependent gauge is the standard one of quantum mechanics in contrast with the case of the time-dependent gauge.

We put forth a criterion on the pair production of bosons and fermions via instantons. A single-instanton is related in a certain way with the single-pair production and multi-instantons with multi-pair production. This criterion implies the no-pair production when there is not any tunneling instanton. In the case of a uniform electric field, when all the contributions from multi-instantons and anti-instantons are taken into account, the pair production rate for bosons calculated according to the instanton interpretation recovers the well-known result from the proper time method. By taking the Pauli-blocking into account the pair production rate for fermions is found also to agree with the standard result. Using the instantons obtained in the WKB (adiabatic) approximation we are also able to provide the formula for the pair production rate of bosons and fermions by inhomogeneous electric fields.

As a by-product we are able to show that a static localized magnetic field cannot produce pairs of bosons or fermions. In the case of magnetic fields the space-dependent gauge reduces the Klein-Gordon equation to time-independent Schrödinger equations with potential wells instead of potential barriers of the electric field case. Since there is not a finite instanton, the pair production of bosons or fermions cannot proceed. A possible infinite instanton gives the zero probability for pair production. Therefore we may conclude that any static magnetic field cannot lead to pair production confirming the well-known result from the proper time method. This will resolve the puzzling issue of pair production by a static localized magnetic field in the canonocal method [14].

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REFERENCES

[1] W. Heisenberg and H. Euler, Z. Physik **98**, 714 (1936); V. Weisskopf, Kgl. Danske Videnskab. Selskabs. Mat.-fys. Medd. **14**, No. 6 (1936).

[2] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[3] B. S. DeWitt, Phys. Rep. **19**, 295 (1975).

[4] L. Parker, Phys. Rev. Lett. **21**, 562 (1968); Phys. Rev. **183**, 1057 (1969); *ibid.* D **3**, 346 (1971).

[5] V. S. Popov, Soviet Phys. JETP **35**, 659 (1972); C. R. Stephens, Ann. Phys. **193**, 255 (1989); Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper, and E. Mottola, Phys. Rev. D **45**, 4659 (1992).

[6] T. Padmanabhan, Prama-J. Phys. **37**, 179 (1991); R. Srinivasan and T. Padmanabhan, Phys. Rev. D **60**, 024007 (1999).

[7] V. S. Popov, Soviet Phys. JETP **34**, 709 (1972).

[8] R. Brout, R. Parentani, and Ph. Spindel, Nucl. Phys. **B353**, 209 (1991); R. Brout, S. Massar, R. Parentani, and Ph. Spindel, Phys. Rep. **260**, 329 (1995); R. Parentani and S. Massar, Phys. Rev. D **55**, 3603 (1997).

[9] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover Pub., New York, 1964).

[10] S. Coleman, *Aspects of Symmetry* (Cambridge Univ. Press, Cambridge, 1985).

[11] A. I. Nikishov, Nucl. Phys. **B21**, 346 (1970).

[12] V. P. Gusynin and I. A. Shovkovy, J. Math. Phys. **40**, 5406 (1999).

[13] N. Fröman and P. O. Fröman, Nucl. Phys. A **147**, 606 (1970); C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I* (Springer, New York, 1999).

[14] L. Sriramkumar and T. Padmanabhan, Phys. Rev. D **54**, 7599 (1996).